The Binomial Asset Pricing Model

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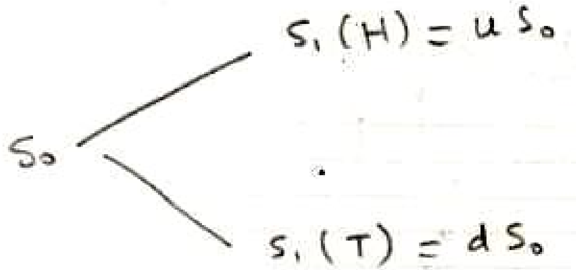


$\Omega = \{H, T\}$

$P\{H\} = p \quad P\{T\} = q = 1-p$

$u = \dfrac{S_1(H)}{S_0} \qquad d = \dfrac{S_1(T)}{S_0}$ Suppose wolg $u > d > 0$

($u = d$ is not interesting)

r = one period interest rate
= risk free borrowing and lending rate

Efficient market $\Rightarrow$ arbitrage is defined as a trading strategy has a positive probability of making money with zero probability of losing money.

In the one-period binomial model, we must assume

$$0 < d < 1+r < u$$ to avoid arbitrage opportunity.

<u>Why?</u> $d > 0$, already assumed. If $d \geq 1+r$, then one can borrow money at risk free rate r, buy one unit of stock,

then at worst get dS_0 at the stock market.

Since $d \geq 1+r$, sell the stock, pay the debt and make money without any risk of losing it.

Borrow \$$S_0$ $\quad -S_0 \qquad$ Debt $\quad -S_0(1+r)$

Buy stock $\quad +S_0 \qquad$ Stock value $\begin{cases} S_0 u \\ S_0 d & \text{worst case} \end{cases}$

$$S_0 d - S_0(1+r) = S_0(d-(1+r)) \geq 0$$

Suppose $u \leq 1+r$

Shortsell stock \$$S_0 \qquad$ Stock value $\begin{cases} S_0 u & \text{(worst case)} \\ S_0 d \end{cases}$

Invest in risk free $-\$S_0 \qquad$ Bank value $\quad (1+r)S_0$

$$(1+r)S_0 - S_0 u = S_0((1+r)-u) \geq 0$$

$\Rightarrow$ Positive probability of making money

We must have $\quad 0 < d < 1+r < u$

Why consider such a simple model:

- arbitrage pricing and risk-neutral pricing is easy to explain
- It is a good approximation to continuous time model
- We can explain fairly complicated theoretical concepts (such as martingales, conditional expectations)

Consider the famous "European Call Option", the owner has the right (not obligation) to buy one share of a stock at time one for the strike price K.

$$S_1(T) < K < S_1(H)$$

If $\omega = \{T\}$ the option is worthless

If $\omega = \{H\}$ the option, when exercised yields $S_1(H) - K$ profit. So, the value of the option is $(S_1(H) - K)^+ = \max\{0, S_1(H) - K\}$. The question is, how much you would be willing to pay for the option?

Let $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$

$S_1(H) = 8$, $S_1(T) = 2$, suppose $K = 5$

Let $X_0 = 1.2$ be the initial wealth, and we buy $\Delta_0 = \frac{1}{2}$ shares of the stock at time 0.

Since $S_0 = 4$, $S_0 \cdot \Delta_0 = 2$, so we borrow 0.8.

Cash position $= X_0 - \Delta_0 S_0 = -0.8$

At time 1, we'll have $(1+r)(X_0 - \Delta_0 S_0) = -1$ (cash)

Stock

$\Delta_0 S_1(H) = 4$

$\Delta_0 S_1(T) = 1$

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = 3$$
$$X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = 0$$

In either case, the value of the option is $(S_1(H)-5)^+ = 3$ or

$$(S_1(T)-5)^+ = 0.$$

Suppose that we sell the option for \$ 1.20. Using the procedure described above we would replicate the option as described above.

	Time 1
Sell option for \$1.20	$X_1(H) = 3$
Buy $\frac{1}{2}$ shares of the stock	$X_1(T) = 0$
Borrow 0.8	

In either case we can close the option. The initial wealth \$ 1.20 is the no-arbitrage price of the option at time zero.

Why?

If we could sell the option for \$ 1.20 + ϵ $(\epsilon > 0)$, then we could invest ϵ in the bank and obtain $\epsilon(1+\frac{1}{4}) = \frac{5}{4}\epsilon = 1.25\,\epsilon$ net worth without any risk. If we sell it for $1.20-\epsilon$, the buyer can do the reverse of the replication above:

Short sell one half shares of the stock → get \$ 2

Use that to buy the option

remaining $= 2-(1.20-\epsilon) = 0.8 + \epsilon$

invest in money market

invest in money market

At time 1 $\nearrow$ H: Option value is $(8-5)^+ = 3$; 0.8 grows into \$1 $\Big)$ pay $\frac{1}{2}$ shares of the stock $= \$4$

$\searrow$ T: option worthless; $0.8 \xrightarrow[(1+r)]{} \$1$ $\Big)$ pos $\frac{1}{2}$ shares of the stock $= \$1$

$\epsilon \cdot 1.25$ = net profit without risk of loss.

$\Rightarrow$ arbitrage

$\Rightarrow$ If the time zero price of the option is 1.20, then there is no arbitrage.

Assumptions:

1. Shares of stock can be subdivided
2. risk free lending – borrowing rates are equal
3. Purchase price = selling price of the stock (no bid-ask spread)
4. Two possible values
5. Short selling is allowed.

<u>Note</u>: The argument above <u>did not</u> depend on the underlying probability structure p, $1-p$.

In the single period model, define a derivative security as a security paying $V_1(H)$ and $V_1(T)$ at time one.

V_0 = price for the derivative security at time zero.

X_0 = initial wealth

Δ_0 = we buy Δ_0 shares of stock at zero.

$X_0 - \Delta_0 S_0$ = cash position at zero.

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$$
$$= (1+r) X_0 + \Delta_0 (S_1 - (1+r) S_0)$$

We want to choose X_0 and Δ_0 in such a way that $X_1(H) = V_1(H)$, $X_1(T) = V_1(T)$

known known

$$V_1(H) = (1+r) X_0 + \Delta_0 (S_1(H) - (1+r) S_0)$$
$$V_1(T) = (1+r) X_0 + \Delta_0 (S_1(T) - (1+r) S_0)$$

unknown

$$X_0 + \Delta_0 \left(\frac{uS_0}{1+r} - S_0 \right) = \frac{V_1(H)}{1+r} \qquad (\mathrm{I})$$

$$X_0 + \Delta_0 \left(\frac{dS_0}{1+r} - S_0 \right) = \frac{V_1(T)}{1+r} \qquad (\mathrm{II})$$

$$\Rightarrow \quad X_0 + \Delta_0 S_0 \left(\frac{u-1-r}{1+r} \right) = \frac{V_1(H)}{1+r}$$

$$X_0 + \Delta_0 S_0 \left(\frac{d-1-r}{1+r} \right) = \frac{V_1(T)}{1+r}$$

Multiply 2^{nd} eqn with -1 and addition

$$\Delta_0 S_0 \left(\frac{u-\cancel{1}-\cancel{r}-d+\cancel{1}+\cancel{r}}{1+r} \right) = \frac{V_1(H)-V_1(T)}{1+r}$$

$$\boxed{\Delta_0 = \frac{V_1(H)-V_1(T)}{S_0(u-d)} = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}}$$

$$X_0 = \frac{V_1(H)}{1+r} - \left(\frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)} \right) S_0 \left(\frac{u-1-r}{1+r} \right)$$

Multiply (I) with $\tilde{p}$, (II) with $\tilde{q} = 1-\tilde{p}$ and add,

$$X_0 + \Delta_0 \left(\frac{1}{1+r} \left[\tilde{p}\, S_1(H) + \tilde{q}\, S_1(T) \right] - S_0 \right) = \frac{1}{1+r} \left(\tilde{p} V_1(H) + \tilde{q} V_1(T) \right)$$

If we choose $\tilde{p}$ in such a way that

$$\frac{1}{1+r}\left(\tilde{p}\, S_1(H) + \tilde{q}\, S_1(T)\right) = S_0 \quad \text{then,}$$

$$X_0 = \frac{1}{1+r}\left(\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)\right)$$

Using $\frac{1}{1+r}\left(\tilde{p}\, u S_0 + \underset{\underset{1-\tilde{p}}{\shortparallel}}{\tilde{q}}\, d S_0\right) = S_0$

$$1+r = \tilde{p}\, u + (1-\tilde{p})d = \tilde{p}(u-d) + d$$

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1-r}{u-d}$$

So, by choosing $\tilde{p}$ and $\tilde{q}$ as above, we can find

$$X_0 = V_0 = \frac{1}{1+r}\left(\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)\right)$$

and $\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$ = delta hedging

<u>Interpret V_0</u> : $u > d$, $d < 1+r \Rightarrow \tilde{p} > 0$

$$\tilde{q} = \frac{u-1-r}{u-d} \qquad u > d, \quad u > 1+r \Rightarrow \tilde{q} > 0$$

$$\Rightarrow \tilde{p} < 1$$

$0 < \tilde{p} < 1$ $\quad \tilde{p} + \tilde{q} = 1$, they are like probabilities

$$V_0 = \frac{1}{1+r}\left\{\underbrace{\tilde{p}\,V_1(H) + \tilde{q}\,V_1(T)}_{\text{expected value of the contingent claim under } \tilde{p} \text{ and } \tilde{q}}\right\} = \text{risk-neutral pricing formula}$$

$\frac{1}{1+r}$: present value

$\tilde{p}$, $1-\tilde{p}$ are called "risk-neutral" probabilities.

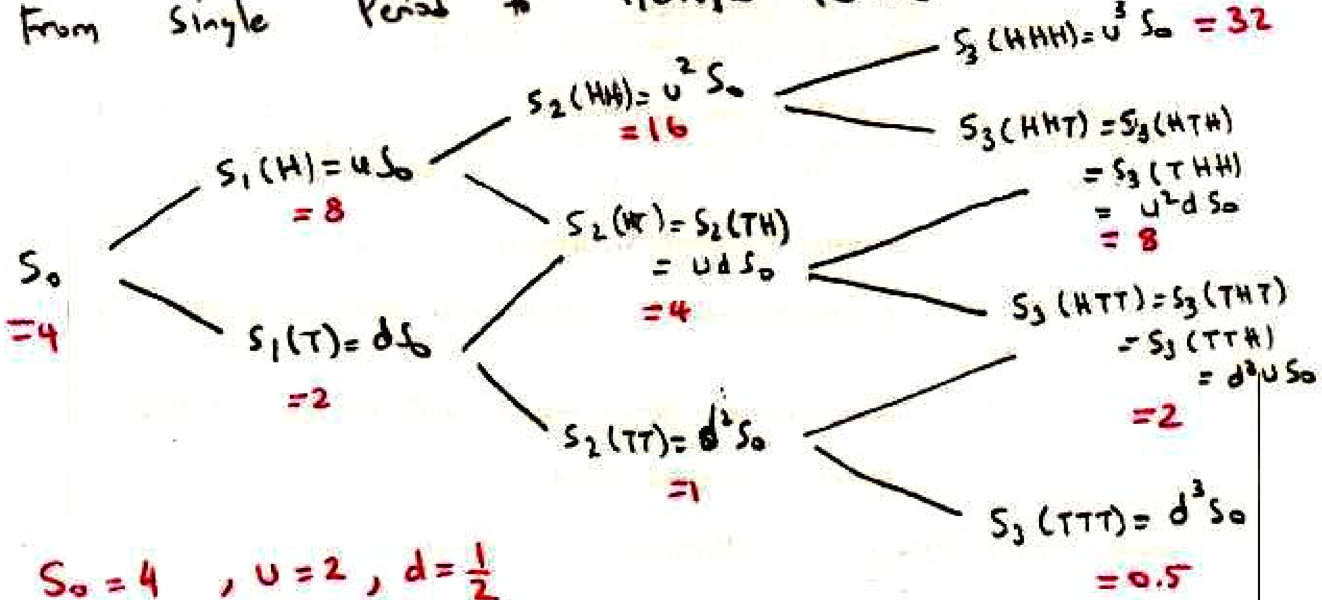
$\tilde{p}$ and $\tilde{q}$ has nothing to do with up or down probabilities! In fact it should be:

$$S_0 < \frac{1}{1+r}\left(p\,S_1(H) + q\,S_1(T)\right)$$

whereas:

$$S_0 = \frac{1}{1+r}\left(\tilde{p}\,S_1(H) + \tilde{q}\,S_1(T)\right)$$

From Single Period to Multiple Periods



$S_0 = 4$, $u = 2$, $d = \frac{1}{2}$

Δ_0 $\qquad$ Δ_1

Consider the call option that gives the owner the right to buy one share of stock for $K at time 2.

$$V_2 = (S_2 - K)^+$$

V_0 = option value at time zero (to be determined)

$$X_1 = \Delta_0 S_1 + (1+r)(V_0 - \Delta_0 S_0)$$ (sell at V_0, buy stocks and invest in money market)

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) \qquad \text{I}$$

$$X_1(T) = \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) \qquad \text{II}$$

$$V_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1)$$

$$V_2(HH) = \Delta_1(H) S_2(HH) + (1+r)\big(X_1(H) - \Delta_1(H) S_1(H)\big) \qquad \text{III}$$

$$V_2(HT) = \Delta_1(H) S_2(HT) + (1+r)\big(X_1(H) - \Delta_1(H) S_1(H)\big) \qquad \text{IV}$$

$$V_2(TH) = \Delta_1(T) S_2(TH) + (1+r)\big(X_1(T) - \Delta_1(T) S_1(T)\big) \qquad \text{V}$$

$$V_2(TT) = \Delta_1(T) S_2(TT) + (1+r)\big(X_1(T) - \Delta_1(T) S_1(T)\big) \qquad \text{VI}$$

6 equations in 6 unknowns $X_1(H), X_1(T), V_0, \Delta_0, \Delta_1(H), \Delta(T)$

Use last 2 equations:

$$\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}$$

Substitute in (V) or (VI):

$$V_2(TT) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} S_2(TT) + (1+r)\left(X_1(T) - \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} S_1(T)\right)$$

Multiply (V) with $\tilde{p}$ and (VI) with $\tilde{q} = 1 - \tilde{p}$

$$\tilde{p} V_2(TH) + \tilde{q} V_2(TT) = \Delta_1(T) \left(\tilde{p} S_2(TH) + \tilde{q} S_2(TT) \right)$$

$$+ (1+r) \left(X_1(T) - \Delta_1(T) S_1(T) \right)$$

$$\tilde{p} = \Delta_1(T) \left(\tilde{p} S_2(TH) + \tilde{q} S_2(TT) - (1+r) S_1(T) \right)$$

$$+ (1+r) X_1(T)$$

$$S_0 \left(\tilde{p}\, sud + (1 \tilde{+} \tilde{p}) dd - (1+r) d \right)$$

$$= S_0 d \left(u \tilde{p} + d - \tilde{p} d + (1+r) \right) = 0$$

with $\tilde{p} = \dfrac{1+r-d}{u-d}$ $\quad \tilde{q} = \dfrac{u-1-r}{u-d}$

$$\Rightarrow \quad X_1(T) = \frac{1}{1+r} \left(\tilde{p} V_2(TH) + \tilde{q} V_2(TT) \right)$$

$$= V_1(T)$$

Similarly

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}$$

$$X_1(H) = V_1(H) = \frac{1}{1+r} \left[\tilde{p} V_2(HH) + \tilde{q} V_2(HT) \right]$$

Plug $X_1(H)$ and $X_1(T)$ in (I) and (II).

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \qquad V_0 = \frac{1}{1+r}\left[\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)\right]$$

(1) Find $\Delta_1(H)$, $\Delta_1(T)$, $X_1(H)$, $X_1(T)$

using $\tilde{p}$ and $\tilde{q}$

(2) Find Δ_0 and V_0

Use $V_2(\omega_1\omega_2) = (S_2(\omega_1\omega_2) - K)^+$

<u>Ex:</u> $r = \frac{1}{4}$ $\quad S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $K = 8$

$$\tilde{p} = \frac{1.25 - 0.5}{1.5} = 0.5 \qquad \tilde{q} = 0.5$$

$$\Delta_1(H) = \frac{(16-8)^+ - (4-8)^+}{16-4} = \frac{8}{12} = \frac{2}{3}$$

$$\Delta_1(T) = \frac{0 - 0}{4 - 1} = 0$$

$$X_1(H) = \frac{1}{1.25}\left[0.5\,(8) + (0.5)\cdot 0\right] = 3.2$$

$$X_1(T) = \frac{1}{1.25}\cdot 0 = 0$$

$$\Delta_0 = \frac{3.2 - 0}{8 - 2} = \frac{3.2}{6} \cong 0.53$$

$$V_0 = \frac{1}{1.25}\left[0.5\,(3.2) + 0.5\,(0)\right] = 1.28$$

<u>Replication:</u> Sell option for 1.28, buy 0.53 unit of stock $4 \times 0.53 = 2.12$ (short $= 0.84$) borrow 0.84 from the bank.

$$X_1(H) = (0.53)(8) + (1.25)(1.28 - 2.12)$$

$$= 4.24 - 1.05 \cong 3.2$$

$$X_1(T) = (0.53)(2) + (1.25)(1.28 - 2.12)$$

$$= 0$$

Now, buy $\frac{2}{3}$ units of stock $\left(\frac{2}{3} \times 8 = \frac{16}{3} \cong 5.33\right.$, borrow $5.33 - 3.2 = 2.13$ from bank$\left.\right)$

$$V_2(HH) = \frac{2}{3}.16 + (1.25)(-2.13)$$

$$= 10.66 - 2.6625 = 8$$

Buyer exercises the option, you get \$8 from her, and sell stock at \$16.

3 stochastic processes (Δ_0, Δ_1)

$$(X_0, X_1, X_2)$$

$$(V_0, V_1, V_2)$$

Theorem: Consider N period binomial asset pricing model with $0 < d < 1+r < u$ and with

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1-r}{u-d}$$

V_N be a r.v. (contingent claim, derivative security)

V_N is a function of $\omega_1 \omega_2 \omega_3 \cdots \omega_N$

$$V_n(\omega_1 \omega_2 \cdots \omega_n) = \frac{1}{1+r}\Big[\tilde{p}\, V_{n+1}(\omega_1 \omega_2 \cdots \omega_n H) + \tilde{q}\, V_{n+1}(\omega_1 \omega_2 \omega_2 \cdots \omega_n T)\Big]$$

$$\Delta_n(\omega_1 \omega_2 \cdots \omega_n) = \frac{V_{n+1}(\omega_1 \omega_2 \cdots \omega_n H) - V_{n+1}(\omega_1 \omega_2 \cdots \omega_n T)}{S_{n+1}(\omega_1 \omega_2 \cdots \omega_n H) - S_{n+1}(\omega_1 \omega_2 \cdots \omega_n T)}$$

$n = 0, 1, \ldots, N-1$

Set $X_0 = V_0$, and define recursively $X_1, X_2, \ldots, X_N$ by $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$, then

$$X_N(\omega_1 \omega_2 \cdots \omega_N) = V_N(\omega_1 \omega_2 \cdots \omega_N) \qquad \omega_1 \omega_2 \cdots \omega_N$$

Ex: $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$ $r = 0.25$ $\tilde{p} = \tilde{q} = 0.5$

Pays $V_3 = \max_{0 \le n \le 3} S_n - S_3$ "lookback option"

$V_3(HHH) = 32 - 32 = 0$

$V_3(HHT) = S_2(HH) - S_3(HHT) = 16 - 8 = 8$

$V_3(HTH) = S_1(H) - S_3(HTH) = 8 - 8 = 0$

$V_3(HTT) = S_1(H) - S_3(HTT) = 8 - 2 = 6$

$V_3(THH) = 0$ $V_3(THT) = 2$ $V_3(TTH) = 2$ $V_3(TTT) = 3.5$

$V_2(HH) = \frac{1}{1.25}\left[0.5(0) + 0.5(8)\right] = 3.2$

$V_2(HT) = \frac{1}{1.25}\left[0.5(0) + 0.5(6)\right] = 2.4$

$V_2(TH) = 0.8$ $V_2(TT) = 2.2$

$V_1(H) = \frac{1}{1.25}\left[0.5(3.2) + 0.5(2.4)\right] = 2.24$

$V_1(T) = 1.2$

$V_0 = \frac{1}{1.25}\left[0.5(2.24) + 0.5(1.2)\right] = 1.376$

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{2.24 - 1.2}{8 - 2} = 0.1733$$